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TURBULENT DISPERSION OF PARTICLES USING EDDY INTERACTION MODELS

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Abstract—In the present paper, the performance of Lagrangian eddy interaction models in homogeneous isotropic stationary turbulence (HIST) is investigated. The relationships between the eddy time and length scales and certain integral scales in HIST are investigated in detail. One-dimensional dispersion of non-fluid particles is then analysed. The influence of the fluid velocity auto-correlation along the particle path, $G(\tau)$, is noted. An upper bound is determined for $G(\tau)$ and is used to investigate the performance of eddy interaction models in the prediction of dispersion of heavy, non-fluid particles. Numerical simulations are used to confirm the analysis for a wide range of particle 'relaxation times'.

Key Words: particle dispersion, eddy interaction model, stationarity, homogeneity, integral scales, numerical simulation, Lagrangian particle tracking

1. INTRODUCTION

This paper discusses aspects of the eddy interaction model, a "Lagrangian" particle tracking method widely used in simulations of turbulent particulate flows. In the eddy interaction model, the motion of a dispersed particulate phase in a turbulent primary flow is determined by particle interactions with a succession of eddies in which the fluid velocity is constant within a finite volume, characterized by the "eddy length" and for a finite time, characterized by the "eddy lifetime". Eddy characteristics are determined from knowledge of the fluid turbulence.

Numerous models have appeared in the literature, following the initial paper of Hutchinson *et al.* (1971), who considered uni-directional radial motion of particles in a turbulent pipe flow. Brown & Hutchinson (1979) developed the model further to consider two-dimensional motion, but retained the discrete eddy velocity specification of the original paper. James *et al.* (1980) used a continuous distribution of eddy velocities in two dimensions to predict particle tracks in a pipe cross-section. Wilkes *et al.* (1982) extended the model of James *et al.* (1980) to three dimensions to determine the influence of axial flow down a pipe. All of the above used eddy characteristic scales which remained constant throughout the flow.

Kallio & Reeks (1989) predicted particle motion in a non-homogeneous turbulent wall layer with eddy characteristics determined from similarity laws. The model has also been developed to account for more complex non-homogeneous turbulence by determination of eddy scales from a numerical simulation of the turbulent primary flow. This coupled approach has been used by Gosman & Ioannides (1981), Faeth and co-workers (see Faeth 1983), Weber *et al.* (1984), Govan *et al.* (1989), Call & Kennedy (1992) and Sommerfeld *et al.* (1992), amongst others, to predict particle motion in various complex turbulent flows. Similar models are also used in commercial flow codes such as CFDS-FLOW3D (AEA Technology 1994).

The approach is therefore widely used in turbulent particulate flows, although there have recently appeared in the literature several other approaches to particle dispersion in such flows (Berlemont *et al.* 1990; Burry & Bergeles 1993; Lu *et al.* 1993, for example). The attraction of the eddy interaction model lies in its conceptual simplicity and the fact that the only statistics required by the model are representative length, time and velocity scales, whereas, in the alternative models, the forms of either the temporal (Lagrangian) or the spatial (Eulerian) velocity auto-correlation functions (or both of these) are required. Recently, Wang & Stock (1992) have shown that the choice of eddy lifetime distribution in the eddy interaction model determines the form of the

Lagrangian fluid velocity auto-correlation function. By considering the idealized case of homogeneous, isotropic and stationary turbulence, it will be shown below that the Eulerian spatial correlations are determined in eddy interaction models by the choice of the eddy length distribution. Knowledge of the temporal and spatial correlations for the eddy interaction model should aid proper comparisons between these models and the more sophisticated models mentioned above. The results of the analysis also have implications for non-homogeneous turbulence and these implications will be discussed where appropriate.

In section 2, the relationship between eddy scales and certain integral scales characteristic of the turbulence is determined in the general case where the eddy velocity, length and lifetime are all random variables. In section 3, the performance of the eddy interaction model in predicting the dispersion of non-fluid particles in HIST is then analysed. The analysis is again performed for the very general case of random eddy velocity, eddy length and eddy lifetime. It is shown that the long-time dispersion coefficient for heavy, non-fluid particles does not exceed the long-time dispersion coefficient for fluid particles. Finally, in section 4, numerical simulations are used to confirm the validity of the analysis for particular cases of the model.

2. TIME AND LENGTH SCALES IN EDDY INTERACTION MODELS

2.1. Eddy interaction models

The main features of eddy interaction models are illustrated in figure 1. At the beginning ($t = 0$) of an interaction, a non-fluid particle, with velocity u_{p0}, is coincident with a fluid particle at the centre of an eddy. Within the eddy, the instantaneous fluid velocity u_e is given by the sum of a mean part and a fluctuating part

$$u_e = U_e + u_e' \qquad [1]$$

Without loss of generality, the mean velocity U_e can be assumed to be zero in homogeneous turbulence. Each component of the fluctuating velocity u_e' is found from a probability distribution with mean zero and variance u_0, the "eddy velocity scale". The characteristic feature of the eddy interaction model, as opposed to other particle dispersion models, is that, throughout the duration of the interaction, u_e remains constant in space and time within the eddy. In general, $u_{p0} \neq u_e$.

At some later time, $t \neq 0$, both the eddy (and with it the fluid particle) and the non-fluid particle will have moved in space. The eddy translates with the instantaneous fluid velocity. However, the fluid and non-fluid particles do not, in general, remain coincident, since the non-fluid particle has a velocity different from that of the fluid. The non-fluid particle remains under the influence of the current eddy until either (i) t exceeds t_e (the eddy lifetime) or (ii) the separation of the fluid and non-fluid particles d(t) exceeds l_e (the eddy length). The time taken to cross the eddy, t_c, is found

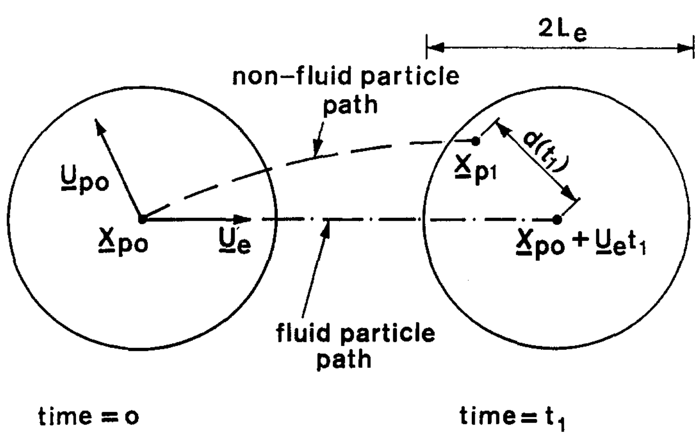

Figure 1. Eddy interaction model.

from the relationship $d(t_c) = l_e$. If either of conditions (i) or (ii) is satisfied, a new eddy is entered. The particle thus remains within an eddy for an interaction time t_i which is the minimum of t_e and t_c.

Particle motions in eddy interaction models are therefore determined by three parameters: (i) eddy velocity, (ii) eddy lifetime and (iii) eddy length. Note that, in the most general case, the eddy lifetime and the eddy length, as well as the eddy velocity, are all random variables. Specification of eddy lifetimes and eddy lengths is discussed in sections 2.2 and 2.3 below. The eddy velocity scale u_0 is set equal to the root mean square of the fluid velocity fluctuations, u', which is assumed to be known.

Eddy scales used by Hutchinson *et al.* (1971), James *et al.* (1980) and Wilkes *et al.* (1982) were obtained from empirical correlations using Laufer's (1954) pipe flow data. Eddy scales in the near-wall turbulence studied by Kallio & Reeks (1989) were determined from laws of similarity at the wall. Gosman & Ioannides (1981), Faeth (1983) and Govan *et al.* (1989) determined eddy scales from a numerical primary flow solution using the k–ε turbulence model. Call & Kennedy (1992) accounted for anisotropic turbulence by the use of a Reynolds stress turbulence model for the primary flow, thereby allowing for different eddy velocities in different coordinate directions.

The purpose of this paper is to investigate the performance of eddy interaction models in HIST. By establishing relationships between eddy scales and those of the flow itself, the models are constrained to function in the limiting cases studied below. The results obtained are expected to lead to an improved understanding of the performance of eddy interaction models in more general turbulent flows. In order to simplify the discussion, the following analysis is restricted to one spatial dimension only.

2.2 Determination of the eddy lifetime—diffusion of fluid particles

To begin with, the case of the diffusion of fluid particles from a point source in HIST is considered. The theory for this process was first given by Taylor (1921), and is discussed in Hinze (1975). The diffusion of fluid particles is determined by the Lagrangian fluid velocity auto-correlation function $R_L(\tau)$. The general form of the Lagrangian fluid velocity auto-correlation function $R_L(t, \tau)$ is

$$R_L(t,\tau) = \frac{\langle u_f(t)u_f(t+\tau)\rangle}{\langle u_f^2(t)\rangle} = \frac{\langle u_f(t)u_f(t+\tau)\rangle}{u'^2} \qquad [2]$$

where and $u_f(t)$ and $u_f(t+\tau)$ are the velocities of a fluid particle at times t and $t+\tau$, respectively, and the angled brackets indicate ensemble averaging over many such particles. In a stationary turbulence, $R_L(t,\tau)$ is independent of the time t and then $R_L(t,\tau) = R_L(\tau)$. $R_L(\tau)$ can be interpreted as a measure of the degree to which the motions of a fluid particle at two different times are correlated (Hinze 1975). For small time differences ($\tau \to 0$), $R_L(\tau)$ is close to 1, whilst for large time differences ($\tau \to \infty$), $R_L(\tau)$ approaches zero.

The Lagrangian integral time scale, τ_L, is defined in terms of the auto-correlation function by

$$\tau_L = \int_0^\infty R_L(\tau)\,d\tau \qquad [3]$$

and is therefore determined completely by $R_L(\tau)$. Note that, in order that τ_L be a valid measure, the underlying turbulence must be stationary. It is assumed that τ_L is a known parameter for the particular case of HIST considered here.

2.2.1. Determination of $R_L(\tau)$ in eddy interaction models. Wang & Stock (1992) have demonstrated that, if the eddy lifetime is distributed randomly according to a probability distribution function (p.d.f.) $f(t_e)$, then a time averaged form for $R_L(\tau)$ is given by

$$R_L(\tau) = \frac{\int_\tau^\infty (t_e - \tau) f(t_e)\,dt_e}{\int_0^\infty t_e f(t_e)\,dt_e} \qquad [4]$$

Integrating by parts, this can be written in the form

$$R_L(\tau) = \frac{\int_\tau^\infty \phi(t_e)\,dt_e}{\int_0^\infty \phi(t_e)\,dt_e} \qquad [5]$$

where

$$\phi(t_e) = \int_{t_e}^\infty f(t')\,dt' \qquad [6]$$

$\phi(t_e)$ is the probability that the (random) eddy lifetime is greater than t_e, i.e. $\phi(t_e) = P$ (eddy lifetime $> t_e$).

In the case of the constant lifetime schemes of Hutchinson *et al.* (1971), Brown & Hutchinson (1979), James *et al.* (1980), Wilkes *et al.* (1982), Gosman & Ioannides (1981) and Govan *et al.* (1979), the p.d.f. is a delta function, $f(t_e) = \delta(t_e - T_e)$, where T_e is the (constant) eddy lifetime. In this case, the auto-correlation assumes the form

$$R_L(\tau) = \begin{cases} 1 - \dfrac{\tau}{T_e}, & \tau \leqslant T_e \\ 0 & \tau > T_e \end{cases} \qquad [7]$$

The Lagrangian integral time scale τ_L^{model} found by integration of [7] is only half of the eddy lifetime T_e, as noted by Wang & Stock (1992) and Minier & Pozorski (1992). Choice of $T_e = 2\tau_L$ ensures "consistency" in the sense that the actual value of Lagrangian integral time scale, τ_L, and that predicted from the model, τ_L^{model}, are the same. Ensuring consistency in this way guarantees that the long-time dispersion of fluid particles is predicted correctly.

In the case (Kallio & Reeks 1989) when the eddy lifetime is distributed according to an exponential probability distribution, $f(t_e) = \exp(-t_e/T)/T$, where T is the mean of the distribution, $R_L(\tau)$ is then found to be

$$R_L(\tau) = e^{-\tau/T} \qquad [8]$$

In this case, $\tau_L^{model} = T$, so that choosing $T = \tau_L$ ensures the consistency mentioned above.

It is noted that Wang & Stock's (1992) method for finding $R_L(\tau)$ is correct when time-averaging and ensemble-averaging are equivalent, i.e. when the "model" turbulence is stationary. In the context of Lagrangian particle tracking methods ensemble-averaging must generally be used. The use of ensemble averaging is typified in the simulations discussed in section 4, in which a large number of identical particles is situated initially, at $t = 0$, at the same location. Each particle undergoes a series of eddy interactions and ensemble-averaged quantities are found by averaging over all particles. Special care must be taken in using eddy interaction models to ensure the equivalence of time-averaging and ensemble-averaging and this is discussed below.

In the eddy interaction model, fluid velocities in successive eddies are assumed to be uncorrelated. Because the fluid velocity within the same eddy is constant in time and space, fluid velocities at different locations in time and space, but within the same eddy, are perfectly correlated. The numerator of [2] is therefore equal to zero for those particles whose velocities at times t and $t + \tau$ are different and is equal to u'^2 for those particles whose velocities are identical. The velocities at times t and $t + \tau$ will be identical if no new eddy has been entered by a particle in the time interval τ. This will be the case if the remainder of the lifetime of the eddy in which the particle resides at time t is greater than τ. $R_L(t, \tau)$ may then be written in the form of the conditional probability

$$R_L(t, \tau) = P(\text{remainder of eddy lifetime} \geqslant \tau \,|\, \text{current time} = t). \qquad [9]$$

In order that ensemble-averaging and time-averaging are equivalent, $R_L(t, \tau)$ must be independent of the starting time t and [5] and [9] must lead to the same expression. From the discussion given above, $R_L(\tau)$ can be written in the form

$$R_L(\tau) = \chi(\tau) \qquad [10]$$

where $\chi(\tau) = P(\text{remainder of lifetime} \geqslant \tau)$. The Lagrangian integral time scale is therefore given by

$$\tau_L = \int_0^\infty R_L(\tau)\,d\tau = \int_0^\infty \chi(\tau)\,d\tau. \quad [11]$$

After integration by parts, this can be written in the form

$$\tau_L = \int_0^\infty \tau f_1(\tau)\,d\tau \quad [12]$$

where $f_1(t)$ is the p.d.f. of the remainder of the eddy lifetime. The Lagrangian integral time scale may therefore by interpreted as the mean remaining lifetime in the eddy interaction model or, equivalently, the mean time to the "next" eddy. Since the remaining lifetime cannot be greater than the full lifetime of the "current" eddy, the mean remaining lifetime cannot exceed the mean eddy lifetime T. Using Wang & Stock's (1992) result that τ_L must exceed $T/2$ gives the following inequality

$$\tau_L \leqslant T \leqslant 2\tau_L. \quad [13]$$

The inequality $\tau_L \leqslant T$ can be re-stated as $\sigma^2 \leqslant T^2$ where σ^2 is the variance of the distribution of eddy lifetimes. Many distributions exist for which the variance exceeds the square of the mean (e.g. some Gamma distributions). It is clear from the analysis that in using such eddy lifetime distributions, time-averaging and ensemble-averaging are not equivalent. The condition that the variance of the lifetime distribution should not exceed the square of its mean therefore provides a necessary condition for a suitable choice of eddy lifetimes and any alternative not satisfying this condition is not "admissable" as an eddy lifetime distribution.

The extremes of inequality [13] in are found (i) in the case where the eddy lifetime is constant, when $2\tau_L = T$ and (ii) in the case where the lifetime is distributed according to an exponential probability distribution, when $\tau_L = T$. The number of computations required by eddy interaction models to simulate fluid particle dispersion over a given time period is dependent on the mean eddy lifetime T and therefore on the choice of the probability distribution which defines the random lifetime. Expression [13] shows that no choice of (admissible) probability distribution requires less interactions than the constant eddy lifetime model and none requires more interactions than the exponential distribution.

By choosing the random lifetime of the first eddy only to come from a distribution such that

$$P(\text{first eddy lifetime} \geqslant t) = \chi(t) \quad [14]$$

where $\chi(t)$ is given by [10], it can be ensured that the model turbulence is in the same statistical state at $t = 0$ as it is $t \to \infty$, i.e. that the turbulence is stationary. For the constant lifetime model, $\chi(t) = 1 - t/T_e$, $0 < t < T_e$, so that the p.d.f. for the lifetime of the first eddy only is $f_1(t) = 1/T_e, 0 < t < T_e$, i.e. the lifetime of the first eddy should be randomly distributed according to a uniform distribution on $[0, T_e]$. Numerical simulations have confirmed the stationarity of this description for the constant lifetime model and have confirmed that the ensemble-averaged auto-correlation is the same as that given by time-averaging using the method of Wang & Stock (1992).

For the constant lifetime model, the stationary state will only be reached if the special treatment noted above is applied to the first eddy. Kallio & Reeks (1989) state that the constant lifetime model is unable to produce a stationary turbulence and is also unable to predict the correct long-time dispersion of fluid particles. The above analysis indicates that simple modifications of the constant lifetime model are able to remove these objections. In the case of the exponential eddy lifetime distribution used by Kallio & Reeks (1989), stationarity is ensured without any special treatment of the first eddy, since it can be shown that the first eddy lifetime should also come from the same exponential distribution as that of subsequent eddies.

2.3. Determination of the eddy length scale—gravitational settling of heavy non-fluid particles

The dispersion of heavy particles falling under the influence of gravity in HIST has been considered by Csanady (1963). In order to analyse this situation, it is necessary to consider the fluid velocity auto-correlation along a particle path, $G(\tau)$. Assuming homogeneity in time and space, $G(\tau)$ may be defined as

$$G(\tau) = \langle v_f(x, t)v_f(x + \tau V_s + \epsilon(\tau), t + \tau)\rangle = u'^2 R_f^p(\tau) \qquad [15]$$

where $v_f(x, t)$ is the fluid velocity at spatial location x and time t, x is the position of the particle at time t, and $x + \tau V_s + \epsilon(\tau)$ is the position at the later time $t + \tau$. V_s is the (constant) settling velocity and $\epsilon(\tau)$ is the change in x due to random velocity fluctuations. $R_f^p(\tau)$ is the fluid velocity auto-correlation along a particle path, normalized so that $R_f^p(0) = 1$.

In the limit of very heavy particles ($\tau V_s \gg \epsilon(\tau)$), the turbulence can be assumed to be "frozen" in time and [15] may be written as

$$G(\tau) = u'^2 R_E(\tau V_s, 0) \qquad [16]$$

or

$$R_f^p(\tau) = R_E(\tau V_s, 0) \qquad [17]$$

Here, $R_E(x, t)$ is the Eulerian fluid velocity auto-correlation, defined by

$$R_E(x, t) = \frac{\langle v_f(x_0, t_0)v_f(x_0 + x, t_0 + t)\rangle}{\langle v_f^2(x_0, t_0)\rangle} = \frac{\langle v_f(x_0, t_0)v_f(x_0 + x, t_0 + t)\rangle}{u'^2} \qquad [18]$$

and is assumed to be independent of t_0 and x_0.

Integrating both sides of [17] with respect to τ, gives

$$\int_0^\infty R_f^p(\tau)\,d\tau = \tau_f^p = \frac{1}{V_s}\int_0^\infty R_e(x, 0)\,dx = \frac{\Lambda_E}{V_s} \qquad [19]$$

where Λ_E is the Eulerian length-scale of the flow and τ_f^p is the integral time scale of the fluid motion following the particle.

2.3.1. Determination of $R_E(x, 0)$ in eddy interaction models. It is usually assumed (Hutchinson *et al.* 1971) that the eddy length is constant throughout the flow. However, it is possible to allow the eddy length to be variable and Burnage & Moon (1990) used an eddy interaction model in which the eddy length was distributed according to the exponential probability distribution. If the length is distributed according to the p.d.f. $g(l_e)$, then the particle crossing time t_c is also random. For a very heavy particle, it can be assumed that the interaction time t_i is determined solely by the crossing time t_c, independently of the eddy lifetime t_e. In this case, it is evident that P(crossing time $> t'$) = P(eddy length $> t'V_s$), so that

$$\phi_f^p(t') = \int_{t'V_s}^\infty g(l_e)\,dl_e, \qquad [20]$$

where $\phi_f^p(t') = P$(crossing time $> t'$).

Using the analysis of Wang & Stock (1992), it follows that

$$R_f^p(\tau) = \frac{\int_\tau^\infty \int_{t'V_s}^\infty g(l_e)\,dl_e\,dt'}{\int_0^\infty \int_{t'V_s}^\infty g(l_e)\,dl_e\,dt'} \qquad [21]$$

Substituting $y = t'V_s$, $x = \tau V_s$, and using [17], we find that

$$R_E(x, 0) = \frac{\int_x^\infty \int_y^\infty g(l_e)\,dl_e\,dy}{\int_0^\infty \int_y^\infty g(l_e)\,dl_e\,dy} = \frac{\int_x^\infty \Psi(y)\,dy}{\int_0^\infty \Psi(y)\,dy} \qquad [22]$$

where $\Psi(y) = P$(eddy length $> y$).

Particle dispersion in the case of heavy particles falling under gravity in HIST is determined by the crossing time of a particle, which is determined by the eddy length distribution. By analogy with the case of fluid particle dispersion in HIST, stationarity in the case of heavy particles falling under gravity can be assured by choosing the crossing time distribution at $t = 0$ to be the same as the distribution of the time to the "next" eddy in the stationary state. It is clear that stationarity in this case corresponds to spatial homogeneity of the turbulence. Spatial homogeneity can therefore be ensured by choosing the first length from a distribution such that $P(\text{first eddy length} > x) = R_E(x, 0)$.

The integral length scale of the model turbulence is given by

$$\Lambda_E^{\text{model}} = \int_0^\infty R_E(x, 0)\,dx \tag{23}$$

Consistency between model and reality is ensured by making the Eulerian integral length scale Λ_E^{model} equal to the actual value of Λ_E. For example, using an eddy model with a constant eddy length L_e, then $g(l_e) = \delta(l_e - L_e)$. In this case

$$R_E(x, 0) = \begin{cases} 1 - \dfrac{x}{L_e}, & x \leqslant L_e \\ 0 & x > L_e \end{cases} \tag{24}$$

Integrating $R_E(x)$, it is found that $\Lambda_E^{\text{model}} = L_e/2$. Therefore, in the same way that in the constant lifetime model, the eddy lifetime T_e should correspond to twice the Lagrangian integral time scale τ_L of the turbulence, for the constant eddy length model, the eddy length L_e should equate to twice the Eulerian length-scale Λ_E. This fact has also been pointed out by Minier & Pozorski (1992). Spatial homogeneity is ensured in this case by specifying that the length of the first eddy should come from a uniform distribution on $[0, L_e]$.

2.4. *Implications for eddy interaction models in non-homogeneous turbulence*

The analysis given above clearly has implications for the length and time scales chosen as representative scales in non-homogeneous turbulence. In particular, the widely-used model of Gosman & Ioannides (1981) assumes that the eddy length and lifetime are equal to the "dissipation scales" L_ϵ and T_ϵ, given by

$$L_\epsilon = C_\mu^{3/4}\frac{k^{3/2}}{\epsilon}, \quad T_\epsilon = \sqrt{3/2}\,C_\mu^{3/4}\frac{k}{\epsilon} \tag{25}$$

where k is the turbulence kinetic energy, ϵ is its rate of dissipation and C_μ is a constant appearing in the k–ϵ turbulence model. If these dissipation scales are considered to be representative scales in the case of non-homogeneous turbulence in the same way as the integral length and time scales in HIST, it is clear that the eddy length and lifetime should in fact be double the values given in [25]. Models using the length and time scales given in [25] should be expected to under-estimate particle dispersion.

3. DISPERSION OF NON-FLUID PARTICLES IN HIST

3.1. *Particle motion*

Non-fluid particles will not, in general, follow exactly the fluctuations of the fluid velocity in turbulent flow. Instead, the particles respond to the balance of forces acting on them. Here, it is assumed that the concentration of particles is small, so that particle–particle interactions can be ignored. In addition, it is assumed that the size of the particles is small compared with the Kolmogorov scale of the turbulence and that they do not spin, so that there are no lift forces on the particles. Under these circumstances, the equation of motion of a solid, spherical particle in a non-uniform flow is given by Maxey & Riley (1983). In the absence of any body forces and under the assumption that the ratio of the density of particles to that of the fluid is large, the only force acting on particles is that due to drag. The equation of motion may then be written in the form

$$\frac{du_p(t)}{dt} + \frac{u_p(t)}{\tau_r} = \frac{u_f(t)}{\tau_r} \tag{26}$$

where τ_r is the relaxation time of the particle, defined by

$$\tau_r = \frac{4}{3}\frac{\rho_p d_p}{\rho_f |u_r| C_d} \qquad [27]$$

where ρ_f and ρ_p are the densities of the fluid and the non-fluid particle, respectively, d_p is the diameter of the non-fluid particle, u_r is the fluid-particle relative velocity and C_d is the drag coefficient.

The product of τ_r and the particle velocity is a measure of the stopping distance if the particle were moving in a stagnant fluid (Young & Hanratty 1991). For small values of τ_r, the particle motion responds quickly to the fluid motion and therefore follows the fluid velocity fluctuations closely. For high values of τ_r, on the other hand, the particle is slow to respond to the fluctations of the fluid flow and does not follow the fluid so closely.

In general, τ_r is dependent upon the particle Reynolds number, $\mathrm{Re}_p = \rho_f |u_r| d_p / \mu_f$, where μ_f is the fluid viscosity. In the case of Stokesian drag, when Re_p is very small ($\ll 1$), the drag coefficient C_d is equal to $24/\mathrm{Re}_p$. In this case, the relaxation time can be written as

$$\tau_r = \frac{\rho_p d_p^2}{18 \mu_f} \qquad [28]$$

The relaxation time is then dependent only on the material properties of the particles and fluid, and does not depend on the particle Reynolds number.

In order to progress with the analysis, the assumption is made that the drag is Stokesian. This has the advantage that the resulting particle motions can be evaluated analytically and also allows direct comparison to be made between the results of this analysis and certain others found in the literature.

3.2. Analysis of particle dispersion

The dispersion of non-fluid particles in turbulent flows is determined by the fluid velocity along the particle track. Pismen & Nir (1978) have considered the following correlations of particle and fluid velocities:

$$H(\tau) = \lim_{t \to \infty} \langle u_p(t) u_p(t + \tau) \rangle \qquad [29]$$

$$G(\tau) = \lim_{t \to \infty} \langle u_f^p(t) u_f^p(t + \tau) \rangle = u'^2 \lim_{t \to \infty} \frac{\langle u_f^p(t) u_f^p(t + \tau) \rangle}{u'^2} = u'^2 R_f^p(\tau) \qquad [30]$$

where $u_p(t)$ and $u_p(t + \tau)$ are the velocities of a non-fluid particle at times t and $t + \tau$, respectively, and the angled brackets indicate ensemble averaging over a large number of such particles. $u_f^p(t)$ and $u_f^p(t + \tau)$ are the velocities of fluid particles situated along the particle track at times t and $t + \tau$, respectively. $H(\tau)$ is thus the non-fluid particle velocity auto-correlation, whilst $G(\tau)$ is the fluid velocity auto-correlation evaluated along the track of a non-fluid particle. $R_f^p(\tau)$ is again the fluid velocity auto-correlation along a particle track, normalized so that $R_f^p(0) = 1$.

Pismen & Nir (1978) showed that, assuming a Stokesian drag law,

$$H(\tau) = \frac{\beta}{2} \int_{-\infty}^{\infty} e^{-\beta |t - \tau|} G(t)\, dt \qquad [31]$$

where $\beta = 1/\tau_r$ is the reciprocal particle relaxation time.

The dispersion of non-fluid particles is determined by $H(t)$. The mean-squared displacement of particles situated initially at $x = 0$ is determined by the equation

$$\bar{X}_p^2(t) = 2 \int_0^t \int_0^{t'} H(\tau)\, d\tau\, dt' \qquad [32]$$

and Pismen & Nir (1978) evaluate the dispersion coefficient for non-fluid particles by means of the relationship

$$D_p(t) = \frac{1}{2} \frac{d}{dt} (\bar{X}_p^2(t)) = \int_0^t H(\tau)\, d\tau \qquad [33]$$

The long-time dispersion coefficient $\bar{D}_p$ is defined as the limiting value of $D_p(t)$ as $t \to \infty$:

$$\bar{D}_p = \int_0^\infty \frac{\beta}{2} \int_{-\infty}^{\infty} e^{-\beta|t-\tau|} G(t)\,dt\,d\tau \quad [34]$$

It follows (Hinze 1975) that

$$\bar{D}_p = \int_0^\infty G(t)\,dt = \int_0^\infty u'^2 R_f^p(t)\,dt = u'^2 \tau_f^p \quad [35]$$

where τ_f^p is the integral time-scale of the fluid velocity evaluated along the path of a non-fluid particle.

3.3. Correlation $G(\tau)$ in eddy interaction models

To evaluate the functions $G(\tau)$ implied by eddy interaction models, the methods of Wang & Stock (1992) can be used. These authors consider the diffusion of fluid particles. In section 2.2.1, it was shown that if the eddy lifetime t_e is distributed randomly accordingly to the p.d.f. $f(t_e)$, then the Lagrangian auto-correlation function is given by

$$R_L(\tau) = R_f^f(\tau) = \frac{\int_\tau^\infty \phi(t_e)\,dt_e}{\int_\tau^\infty \phi(t_e)\,dt_e} \quad [36]$$

where $\phi(t') = P(\text{lifetime} > t')$ and $R_f^f(\tau)$ denotes the fluid velocity auto-correlation along the path of a fluid particle.

Consider now the motion of non-fluid particles in eddy interaction models. The interaction time t_i, defined as the time spent by a particle within an eddy, is determined as follows:

(1) if $|u_r| \leqslant l_e/\tau_r$ (particle is captured),

$$t_i = t_e,$$

(2) if $|u_r| > l_e/\tau_r$ (particle crosses eddy in finite time t_c),

$$t_i = \min(t_e, t_c),$$

where $t_c = -\tau_r \log(1 - l_e/(|u_r|\tau_r))$ (see section 4.1 below) is the time taken by a particle to travel a distance equal to l_e and where u_r is the (random) fluid/non-fluid particle relative velocity. The interaction time t_i is thus a random variable. Condition (2) above ensures that the interaction time never exceeds the eddy lifetime. The probability distribution of the interaction time is determined below. Knowledge of the distribution enables the methods described in section 2 to be used to determine the fluid velocity correlation along the particle path, $G(\tau) = u'^2 R_f^p(\tau)$.

Given an eddy of length l_e, a particle will not have crossed the eddy within a time t' if the magnitude of the fluid/non-fluid relative velocity $|u_r|$ is less than $u_{max}(t', l_e)$, where

$$u_{max}(t', l_e) = \frac{l_e}{\tau_r(1 - e^{-t'/\tau_r})} \quad [37]$$

If the relative speed is greater than $u_{max}(t', l_e)$, the non-fluid particle will have crossed the eddy in the time t' (i.e. $t_c < t'$).

The interaction time t_i is greater than t' if both the eddy lifetime t_e and the crossing time t_c are greater than t'. Thus, denoting by $\phi_f^p(t')$ the probability that, along a particle track, the interaction time is greater than t', we can write

$$\phi_f^p(t') = \int_{t'}^\infty f(t_e)\,dt_e \int_0^\infty g(l_e) \int_0^{u_{max}(t', l_e)} 2h(u_r)\,du_r\,dl_e \quad [38]$$

where $f(t_e)$ is the p.d.f. of the eddy lifetime, $g(l_e)$ is the p.d.f. of the eddy length and $h(u_r)$ is the p.d.f. of u_r. It has been assumed in the above expression that, without loss of generality, $h(u_r)$ is symmetrical about $u_r = 0$. It is also assumed that the eddy length, lifetime and velocity are all independently distributed. This is consistent with each of the eddy interaction models discussed in section 2, including the models of Hutchinson *et al.* (1971), Gosman & Ioannides (1981), Faeth

(1983), Kallio & Reeks (1989) and Burnage & Moon (1990). The case when the eddy length and eddy lifetime are correlated is discussed at the end of this section.

Using the methods of section 2, the fluid velocity auto-correlation along the track of a non-fluid particle is given by

$$R_{\mathrm{f}}^{\mathrm{p}}(\tau)=\frac{\int_{\tau}^{\infty}\phi_{\mathrm{f}}^{\mathrm{p}}(t')\,\mathrm{d}t'}{\int_{0}^{\infty}\phi_{\mathrm{f}}^{\mathrm{p}}(t')\,\mathrm{d}t'} \quad [39]$$

i.e.

$$R_{\mathrm{f}}^{\mathrm{p}}(\tau)=\frac{\int_{\tau}^{\infty}\int_{t'}^{\infty}f(t_{\mathrm{e}})\,\mathrm{d}t_{\mathrm{e}}\int_{0}^{\infty}g(l_{\mathrm{e}})\int_{0}^{u_{\max}(t',l_{\mathrm{e}})}2h(u_{\mathrm{r}})\,\mathrm{d}u_{\mathrm{r}}\,\mathrm{d}l_{\mathrm{e}}\,\mathrm{d}t'}{\int_{0}^{\infty}\int_{t'}^{\infty}f(t_{\mathrm{e}})\,\mathrm{d}t_{\mathrm{e}}\int_{0}^{\infty}g(l_{\mathrm{e}})\int_{0}^{u_{\max}(t',l_{\mathrm{e}})}2h(u_{\mathrm{r}})\,\mathrm{d}u_{\mathrm{r}}\,\mathrm{d}l_{\mathrm{e}}\,\mathrm{d}t'} \quad [40]$$

It is evident from this expression that

$$G(\tau)=u'^{2}R_{\mathrm{f}}^{\mathrm{p}}(\tau)\geqslant 0 \quad [41]$$

Note that, since a fluid particle will always be captured by an eddy, the interaction time t_{i} for such a particle is always equal to the eddy lifetime t_{e}. In this case, we can write

$$\phi_{\mathrm{f}}^{\mathrm{f}}(t')=\int_{t'}^{\infty}f(t_{\mathrm{e}})\,\mathrm{d}t_{\mathrm{e}} \quad [42]$$

where $\phi_{\mathrm{f}}^{\mathrm{f}}(t')=P(\text{interaction time}>t')=P(\text{eddy lifetime}>t')$.

We next define the function $\rho(t')$ as follows:

$$\rho(t')=\frac{\phi_{\mathrm{f}}^{\mathrm{p}}(t')}{\phi_{\mathrm{f}}^{\mathrm{f}}(t')}=\int_{0}^{\infty}g(l_{\mathrm{e}})\int_{0}^{u_{\max}(t',l_{\mathrm{e}})}2h(u_{\mathrm{r}})\,\mathrm{d}u_{\mathrm{r}}\,\mathrm{d}l_{\mathrm{e}} \quad [43]$$

Since $u_{\max}(t',l_{\mathrm{e}})$ is a decreasing function of t', for any given l_{e}, then $\rho(t')$ is also decreasing in t'. It is shown in appendix A that, if $\rho(t')$ is a decreasing function of t', then the following inequality holds:

$$R_{\mathrm{L}}(\tau)=R_{\mathrm{f}}^{\mathrm{f}}(\tau)\geqslant \mathrm{R}_{\mathrm{f}}^{\mathrm{p}}(\tau) \quad [44]$$

As $\tau_{\mathrm{r}}\rightarrow\infty$, the non-fluid particle becomes stationary and $R_{\mathrm{f}}^{\mathrm{p}}(\tau)$ approaches the Eulerian (fixed-point) auto-correlation $R_{\mathrm{E}}(0,\tau)$. In this case, $u_{\max}(t',l_{\mathrm{e}})$ becomes l_{e}/t' and the Eulerian auto-correlation function can be written as

$$R_{\mathrm{E}}(0,\tau)=R_{\mathrm{f}}^{\mathrm{p}}(\tau)=\frac{\int_{\tau}^{\infty}\phi_{\mathrm{E}}(t')\,\mathrm{d}t'}{\int_{0}^{\infty}\phi_{\mathrm{E}}(t')\,\mathrm{d}t'} \quad [45]$$

where

$$\phi_{\mathrm{E}}(t')=\int_{0}^{\infty}g(l_{\mathrm{e}})\int_{0}^{l_{\mathrm{e}}/t'}2h(u_{\mathrm{r}})\,\mathrm{d}u_{\mathrm{r}}\,\mathrm{d}l_{\mathrm{e}}. \quad [46]$$

If it is assumed that both the fluid velocity and the particle velocity are normally distributed, the p.d.f. of the fluid/non-fluid relative velocity, $h(u_{\mathrm{r}})$, decreases as u_{r} increases. It can be shown that, if $h(u_{\mathrm{r}})$ is a decreasing function of u_{r}, then the ratio $\phi_{\mathrm{E}}(t')/\phi_{\mathrm{f}}^{\mathrm{p}}(t')$ is decreasing in t' and the analysis of appendix A can be used to show that

$$R_{\mathrm{f}}^{\mathrm{p}}(\tau)\geqslant R_{\mathrm{E}}(0,\tau) \quad [47]$$

for all values of τ.

3.3.1. Integral time scales and dispersion coefficients. The auto-correlation functions derived in section 3.3 can be used to determine the integral time scale of the fluid velocities following the path of a non-fluid particle. From [44] and [47], it follows that the integral scale of the fluid velocity following a non-fluid particle must lie between the Lagrangian and Eulerian integral time scales

$$\tau_L \geqslant \tau_p^f \geqslant \tau_E, \qquad [48]$$

where

$$\tau_E = \int_0^\infty R_E(0, \tau)\, d\tau \qquad [49]$$

is the Eulerian integral time scale.

Using [35], it can be seen that, in simulations of particle dispersion using eddy interaction models, the dispersion coefficient of the non-fluid particles is therefore constrained to lie in the range

$$u'^2\tau_L \geqslant \bar{D}_p = u'^2\tau_p^f \geqslant u'^2\tau_E. \qquad [50]$$

It is therefore clear that, assuming Stokesian drag, using eddy interaction models of the kind considered in this analysis, non-fluid particles will disperse less rapidly in HIST, in the long term, than fluid particles. In the next section, in order to test the predictions of the above analysis, a series of numerical tests has been performed to simulate dispersion for various values of τ_r, the particle relaxation time.

The analysis has been completed for the case when the eddy length and lifetime are uncorrelated. However, it is clear that the constraint, even if they are correlated, the interaction time cannot exceed the eddy lifetime that leads to a general reduction of the interaction time compared with the case of fluid particles. This in turn implies that the time to the "next" eddy is likewise decreased. Using the analysis of section 2, the integral time scale of the fluid velocity following a particle can be interpreted as the mean time to the next eddy interaction. The constraint that the interaction time cannot exceed the eddy lifetime leads to the constraint that $\tau_L \geqslant \tau_p^f$ even when the eddy length and lifetime are correlated. In the case of Stokesian drag, this will inevitably lead to the reduced dispersion of non-fluid particles compared with fluid particles, irrespective of the correlation between eddy length and eddy lifetime.

In the case where the drag on particles does not follow Stokes' law, the equation of motion of a non-fluid particle cannot be solved explicitly. The analysis given above remains valid except that it should be noted that $u_{max}(t', l_e)$, the fluid/non-fluid relative velocity above which a particle crosses an eddy within the time t', cannot be found analytically. Under any drag law, $u_{max}(t', l_e)$ continues to be decreasing as t' increases so that the function defined by [43] is again decreasing as t' increases. Inequality [44] therefore remains valid even in the case of a non-linear drag law. However, the relationship between the dispersion coefficients given in expression [46] depends upon the assumption of Stokesian drag, and no analogous relationship can be derived in the case of non-Stokesian drag.

4. NUMERICAL MODELLING OF PARTICLE DISPERSION

4.1. *Particle motion*

From the assumption of Stokesian drag, it is possible to derive analytical expressions for the particle velocity and position at the end of an interaction with an eddy. Suppose that, on entry to an eddy, a non-fluid particle is located at X_{p0} and has a velocity of u_{p0}. The fluid velocity within the eddy is u_e (see figure 1), and the interaction time is given by t_i. Remembering that u_e is constant for a single particle–eddy interaction, the equation of motion of the particle can be written as

$$e^{-t/\tau_r}\frac{d}{dt}(u_p(t)e^{t/\tau_r}) = \frac{u_e}{\tau_r} \qquad [51]$$

and the particle position is obtained by integration of the velocity

$$X_p(t) = X_{p0} + \int_0^t u_p(t')\, dt' \qquad [52]$$

The integral in the above expression can be evaluated analytically, so that, after the interaction with the eddy, the updated particle velocity $u_{\rm pl}$ is given by

$$u_{\rm pl} = u_{\rm e} - (u_{\rm e} - u_{\rm p0})e^{-t_{\rm i}/\tau_{\rm r}} \quad [53]$$

The updated particle position $X_{\rm pl}$ is given by

$$X_{\rm pl} = (X_{\rm p0} + t_{\rm i}u_{\rm e}) - (u_{\rm e} - u_{\rm p0})\tau_{\rm r}(1 - e^{-t_{\rm i}/\tau_{\rm r}}) \quad [54]$$

Note that the crossing time $t_{\rm c}$ is given by the solution of the equation

$$|X_{\rm p}(t_{\rm c}) - X_{\rm p0} - t_{\rm c}u_{\rm e}| = l_{\rm e} \quad [55]$$

i.e.

$$t_{\rm c} = -\tau_{\rm r}\log\left(1 - \frac{l_{\rm e}}{|u_{\rm r}|\tau_{\rm r}}\right) \quad [56]$$

where $u_{\rm r}, = u_{\rm e} - u_{\rm p0}$, is the fluid–particle relative velocity. The above expression is valid only if $l_{\rm e}/(|u_{\rm r}|\tau_{\rm r}) < 1$, and if this condition holds, $t_{\rm i}$ is set equal to $t_{\rm c}$. If this inequality does not hold, the particle is "captured", in which case $t_{\rm i}$ is set equal to $t_{\rm e}$. Because the interaction time can be determined prior to the interaction, only one integration is required per eddy.

In the case of a small relaxation time, the particles will almost always be captured by the eddies. The interaction time will thus be equal to the eddy lifetime and the particle velocities will quickly approach the fluid velocities. In this case, the dispersion of non-fluid particles will be similar to that of fluid particles. In the case of a large relaxation time, the interaction time will more often be equal to the particle crossing time. In such cases, the interaction time will be independent of the eddy lifetime. In the limit as the relaxation time tends to infinity, the probability that the interaction time is given by the eddy lifetime approaches zero. In addition, the particle appears to the fluid to be a fixed point, so that the Eulerian fluid velocity auto-correlation determines dispersion for very heavy particles.

4.2. Numerical results

Results are presented in this section for three values (0, 1 and 2) of the ratio $\alpha = u'\tau_{\rm L}/\Lambda_{\rm E}^{\rm model}$ for fixed u' and $\tau_{\rm L}$. α is the ratio of the Lagrangian integral time scale, $\tau_{\rm L}$, to the "eddy turnover time" $\Lambda_{\rm E}/u'$, and is usually assumed to be of the order of 1. The case $\alpha = 0$ corresponds to $G(\tau) = u'^2R_{\rm L}(\tau)$, for which the analytical solution for $H(\tau)$ is given in appendix B. In order to test the validity of the analysis, two models for eddy lifetime are used. In both models the eddy velocity $u_{\rm e}$ was assumed to come from a normal distribution with mean zero and standard deviation u'. All computer simulations used 20,000 particles and each particle was followed for a duration of $1000\,\tau_{\rm L}$. The long-time dispersion coefficients $\bar{D}_{\rm p}$ were evaluated approximately as $\bar{D}_{\rm p} = \bar{X}_{\rm p}^2(t = 1000\,\tau_{\rm L})/(2000\,\tau_{\rm L})$.

In the first model, constant eddy lifetime and constant eddy length are used. The variation of $\bar{X}_{\rm p}^2(t)$ with time can be seen from figure 2, in which $\bar{X}_{\rm p}^2(t)/(2\,u'^2t\tau_{\rm L})$ is plotted against $t/\tau_{\rm L}$ for the three cases $\alpha = 0$, 1, 2. Results for two values of $\tau_{\rm r}/\tau_{\rm L}$, 0.01 and 10, are shown, indicating the behaviour of "fluid" and heavy non-fluid particles. It is evident from figure 2 that, especially for $\alpha = 2$ but also, to a lesser extent, for $\alpha = 1$, whilst fluid particles eventually disperse as $2t\tau_{\rm L}$, the heavy particles are dispersed less rapidly. However, for $\alpha = 0$, heavy particles do eventually disperse as rapidly as fluid particles. The reason for this is that when $\alpha = 0$, the long time dispersion coefficients for fluid and non-fluid particles are equal (see appendix B). For the cases $\alpha = 1$ and $\alpha = 2$, the particle dispersion coefficient is generally less than that for fluid particles, as a result of the inequality $G(\tau) < u'^2R_{\rm L}(\tau)$ for these cases. The results also indicate that for heavy particles, the long-time dispersion coefficient $\bar{D}_{\rm p}$ decreases with increasing α, i.e. with decreasing eddy length.

The second model assumes an exponential distribution for the eddy lifetime, as used by Kallio & Reeks (1989). Results for the cases $\alpha = 0$, 1 and 2 for particle relaxation times $\tau_{\rm r}/\tau_{\rm L} = 0.01$ and 10 are shown in figure 3. It can be seen that, for the heavy particles, dispersion for the cases $\alpha = 1$ and 2 is under-predicted using the exponential model, compared with the constant lifetime model. This is thought to be due to a form of "filtering" whereby, when the exponential

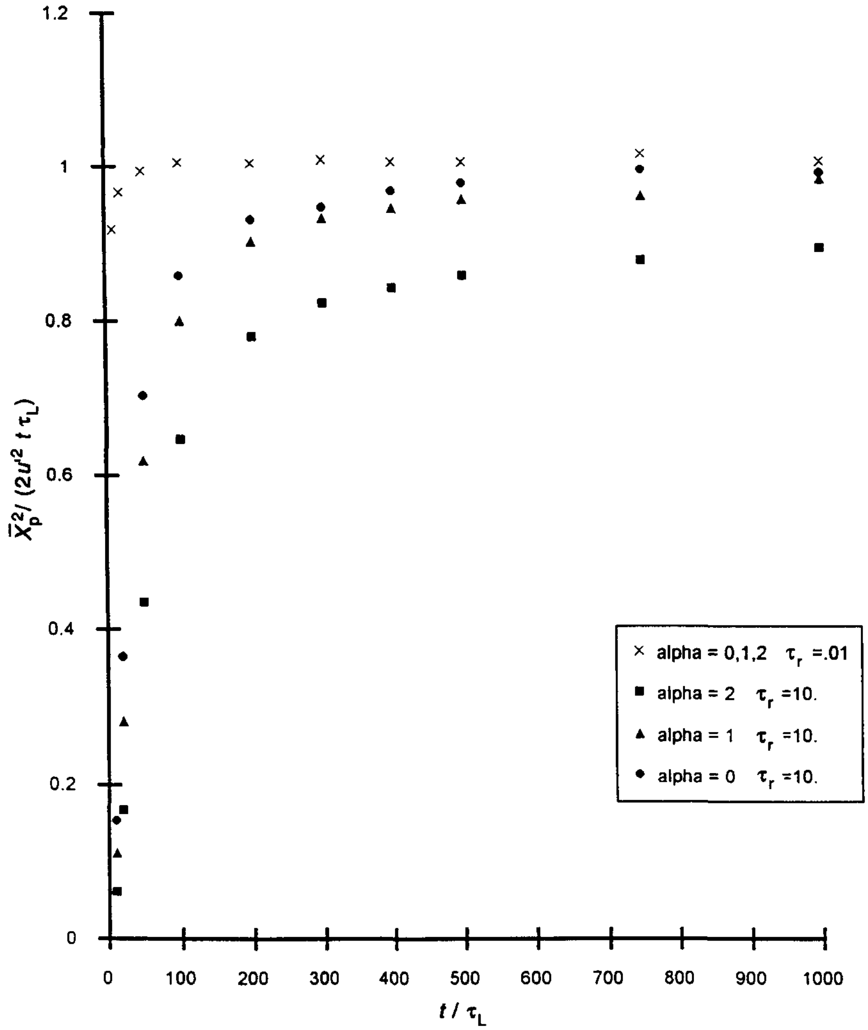

Figure 2. Normalized mean-square displacement vs time for "fluid" and "heavy" particles—constant lifetime model.

distribution happens to produce a large eddy lifetime, there is an increased chance that the eddy time may be determined by the crossing time. The contribution from large lifetimes may well be curtailed, tending to lead to less dispersion than would otherwise occur. Again $\bar{D}_p(\alpha = 0) > \bar{D}_p(\alpha = 1) > \bar{D}_p(\alpha = 2)$ for heavy particles, indicating the decrease in dispersion coefficient with decreasing eddy length. It is also noted that, for the exponential distribution, the mean eddy lifetime is equal to τ_L, whereas, for the constant lifetime model, it is $2\tau_L$. It follows that approximately twice as many eddy interactions take place in a given period of time in the exponential model compared with the constant lifetime model.

The ratio of long-time dispersion coefficients for various values of α and τ_r/τ_L for both models is plotted in figure 4, which also illustrates a comparison with the analytical results of Pismen & Nir (1978). These authors predict that heavy particles disperse more rapidly, in the long-time limit, than fluid particles. The numerical results of this section show that this phenomenon is not observed using conventional eddy interaction methods, confirming the analysis of the previous section. The experiments of Wells & Stock (1983) and the direct numerical simulation results of Squires & Eaton (1991) indicate that it may be possible that heavy particles disperse more rapidly, in the long term, than fluid particles. Modifications of eddy interaction models which lead to the capability of modelling such a phenomenon are to be investigated in a subsequent paper (Graham 1996).

5. CONCLUSIONS AND FINAL COMMENTS

This study of eddy interaction models for the dispersion of fluid and non-fluid particles in homogeneous, isotropic and stationary turbulence leads to the following conclusions.

(1) Following Wang & Stock (1992), time-scale consistency has been ensured by a suitable choice of the eddy lifetime distribution. This distribution has been chosen so that the Lagrangian integral time scale of the model turbulence is equal to that of the actual turbulence. The relationship between time-averaging and ensemble-averaging in eddy interaction models has been investigated and leads to a simple interpretation of the Lagrangian integral time scale in these models. In addition, a method has been introduced which, by considering the lifetime of the first eddy only, forces the model turbulence to be stationary. These measures ensure that the long-time dispersion of fluid particles is predicted correctly.

(2) The relationship between the eddy length distribution and the Eulerian spatial velocity auto-correlation has been derived. A method has been introduced to ensure consistency of length scales by a suitable choice of the eddy length distribution. This distribution must be chosen so that the Eulerian integral length scale of the model turbulence is equal to that of the actual turbulence. By considering the length of the first eddy, a method has been

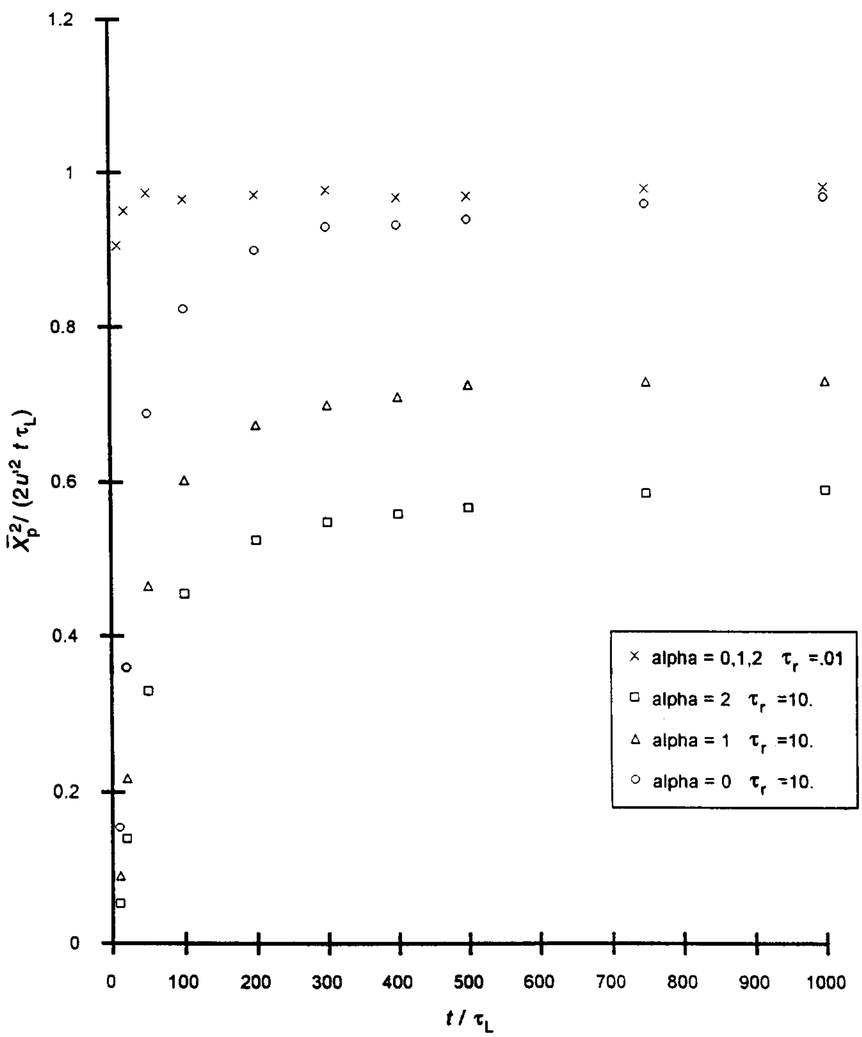

Figure 3. Normalized mean-square displacement vs time for "fluid" and "heavy" particles—exponential lifetime model.

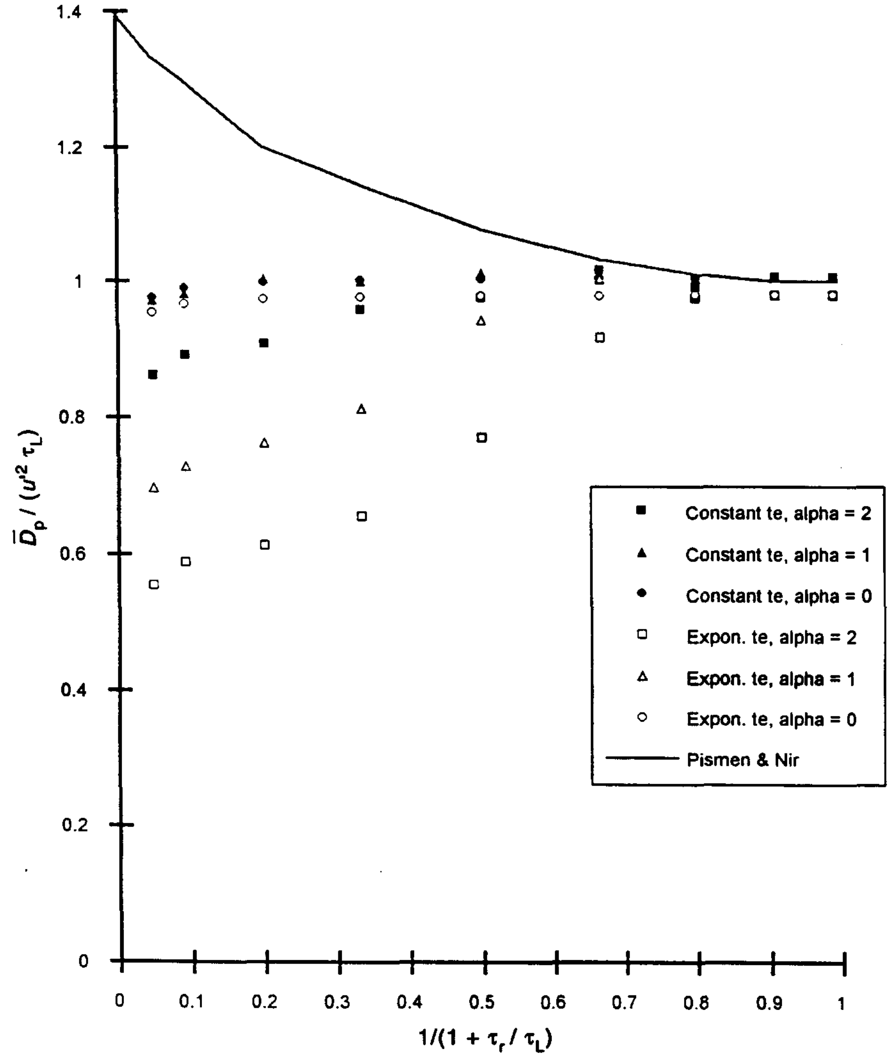

Figure 4. Long-time dispersion coefficients for various relaxation times—both models.

introduced to enforce spatial homogeneity of the model turbulence. These measures ensure that the long-time dispersion of heavy particles settling under gravity is predicted correctly.

(3) Dispersion of non-fluid particles has been analysed. When the drag on a particle follows Stokes' law, this dispersion is determined by the auto-correlation of the fluid velocities following a non-fluid particle, $G(\tau)$. An analytical relationship has been derived between $G(\tau)$ and the eddy length, lifetime and velocity distributions. It has been shown that in conventional eddy interaction models, in which the eddy interaction time never exceeds the eddy lifetime, the integral time scale τ_L^p associated with $G(\tau)$ cannot exceed the Lagrangian integral time scale τ_L. This is the case whether or not the drag on a particle follows Stokes' law. However, in the case where the drag is Stokesian, this fact can also be used to show that the long-time dispersion coefficient of non-fluid particles is always less than that for fluid particles.

(4) Numerical simulations confirm the dispersion analysis for particular cases of the eddy interaction model. All the simulations have used constant eddy length specification. It has been shown that the dispersion coefficient of non-fluid particles is affected by the imposition of finite eddy length L_e and that, for heavy particles, (large τ_r/τ_L), the dispersion coefficient generally decreases with decreasing L_e.

(5) Eddy interaction models utilizing an exponential distribution for the eddy lifetime are shown to be affected more by the imposition of a finite eddy length than the corresponding models

using constant eddy lifetime. In addition, in modelling the dispersion of fluid particles, the number of eddies encountered in using the exponential lifetime scheme is approximately twice that encountered using the constant eddy lifetime scheme, so that each computer simulation takes twice as long. These facts lead to the conclusion that the constant eddy lifetime scheme is to be preferred to the exponential lifetime scheme.

These conclusions enable adjustments to be made to existing eddy interaction models in order to ensure consistency of time and length scales, and lead to a model turbulence which is stationary and homogeneous. More importantly, perhaps, is the conclusion that, for the models discussed, fluid particles will disperse more rapidly, in the long term, than non-fluid particles.

Experimental evidence and independent numerical simulations indicate the possibility that long-time dispersion of non-fluid particles can exceed that of fluid particles. It is clear that conventional eddy interaction models are not capable of predicting such behaviour. Modifications of eddy interaction models which do predict increased dispersion of non-fluid particles are to be reported in a subsequent paper (Graham 1996).

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APPENDIX A

Proof that $R_1(\tau) \geqslant R_2(\tau)$ if $\rho(t') = \frac{[\phi_2(t')]}{[\phi_1(t')]}$ is Decreasing

Suppose that two different eddy interaction time distributions $f_1(t_e)$ and $f_2(t_e)$ lead eventually to two different auto-correlations $R_1(\tau)$ and $R_2(\tau)$.
By definition

$$\phi_1(t') = \int_{t'}^{\infty} f_1(t_e)\,dt_e, \qquad [A1]$$

$$\phi_2(t') = \int_{t'}^{\infty} f_2(t_e)\,dt_e, \qquad [A2]$$

and by hypothesis

$$\rho(t') = \frac{\phi_2(t')}{\phi_1(t')} \quad \text{is decreasing in } t'. \qquad [A3]$$

Using Wang & Stock's method to determine $R_1(\tau)$ and $R_2(\tau)$, it follows that

$$R_1(\tau) - R_2(\tau) = \frac{\int_{\tau}^{\infty} \phi_1(t')\,dt'}{\int_0^{\infty} \phi_1(t')\,dt'} - \frac{\int_{\tau}^{\infty} \phi_2(t')\,dt'}{\int_0^{\infty} \phi_2(t')\,dt'} \qquad [A4]$$

$$= \frac{\int_0^\tau \phi_2(t')\,dt'}{\int_0^\infty \phi_2(t')\,dt'} - \frac{\int_0^\tau \phi_1(t')\,dt'}{\int_0^\infty \phi_1(t')\,dt'} \qquad [A5]$$

$$= \frac{\int_0^\tau \phi_2(t')\,dt' \int_0^\infty \phi_1(t')\,dt' - \int_0^\tau \phi_1(t')\,dt' \int_\tau^\infty \phi_2(t')\,dt'}{\int_0^\infty \phi_1(t')\,dt' \int_0^\infty \phi_2(t')\,dt'} \qquad [A6]$$

Since the denominator is positive, then the sign of $R_1(\tau) - R_2(\tau)$ is the same as the sign of the numerator

$$N = \int_0^\tau \phi_2(t')\,dt' \int_0^\infty \phi_1(t')\,dt' - \int_0^\tau \phi_1(t')\,dt' \int_0^\infty \phi_2(t')\,dt' \qquad [A7]$$

This can be written as

$$N = \int_0^\tau \int_0^\infty \phi_2(t')\phi_1(t'')\,dt''\,dt' - \int_0^\tau \int_0^\infty \phi_1(t')\phi_2(t'')\,dt''\,dt'. \qquad [A8]$$

Substituting $\phi_2(t') = \rho(t')\phi_1(t')$, it follows that

$$N = \int_0^\tau \int_0^\infty \rho(t')\phi_1(t')\phi_1(t'')\,dt''\,dt' - \int_0^\tau \int_0^\infty \rho(t'')\phi_1(t')\phi_1(t'')\,dt''\,dt'. \qquad [A9]$$

$$= \int_0^\tau \int_0^\infty \phi_1(t')\phi_1(t'')(\rho(t') - \rho(t''))\,dt''\,dt'. \qquad [A10]$$

$$= \int_0^\tau \int_\infty^\infty \phi_1(t')\phi_1(t'')(\rho(t') - \rho(t''))\,dt''\,dt'. \qquad [A11]$$

since

$$\int_0^\tau \int_0^\tau \phi_1(t')\phi_1(t'')(\rho(t') - \rho(t''))\,dt''\,dt' = 0 \qquad [A12]$$

Thus, if $\rho(t')$ is decreasing in t', $\rho(t') \geqslant \rho(t'')$ in [A11], so that N is positive. Thus, from [A6],

$$R_1(\tau) - R_2(\tau) \geqslant 0$$

or

$$R_1(\tau) \geqslant R_2(\tau) \qquad [A13]$$

The result holds for all $\tau \geqslant 0$.

APPENDIX B

Particle Velocity Auto-correlation when $G(\tau) = u_0^2 R_L(\tau)$

Equation [31] may be written as

$$H(\tau) = u_0^2 \frac{\beta}{2} \int_{-\infty}^{\infty} e^{-\beta|t-\tau|} R_L(t)\,dt \qquad [B1]$$

Constant lifetime model

Substituting $R_L(\tau) = u_0^2(1 - |\tau|/T_e)$ in [B1] it follows that

$$H(\tau) = u_0^2 \left\{1 - \frac{\tau}{T_e} + \frac{1}{2\beta T_e}(e^{-\beta T_e}(e^{-\beta\tau} + e^{\beta\tau}) - 2e^{-\beta\tau})\right\}, \quad \tau \leqslant T_e$$

$$= \frac{u_0^2}{2\beta T_e} e^{-\beta\tau}(e^{-\beta T_e} + e^{\beta T_e} - 2), \tau \geqslant T_e \qquad [B2]$$

This expression for $H(\tau)$ leads to

$$\bar{D}_{\rm p} = \int_0^\infty H(\tau)\,{\rm d}\tau = u_0^2 \frac{T_{\rm e}}{2} = \bar{D}_{\rm f} \qquad \text{[B3]}$$

i.e. the dispersion coefficient for massive particles is independent of the particles relaxation time and is equal to that of fluid particles, $\bar{D}_{\rm f}$.

It is also noted that

$$H(0) = u_0^2 \left\{ 1 - \frac{1}{\beta T_{\rm e}} (1 - {\rm e}^{-\beta T_{\rm e}}) \right\} \qquad \text{[B4]}$$

where $H(0) = 2/3 \times$ energy of particle fluctuations per unit mass.

Exponential lifetime model

Substituting $R_2(\tau) = {\rm e}^{-|\tau|/\tau_{\rm L}}$ into [B1] leads to

$$H(\tau) = u_0^2 \beta \tau_{\rm L} \frac{\{\beta \tau_{\rm L} {\rm e}^{-\tau/\tau_{\rm L}} - {\rm e}^{-\beta\tau}\}}{((\beta\tau_{\rm L})^2 - 1)} \qquad \text{[B5]}$$

This expression for $H(\tau)$ gives, again,

$$\bar{D}_{\rm p} = \int_0^\infty H(\tau)\,{\rm d}\tau = u_0^2 \tau_{\rm L} = \bar{D}_{\rm f} \qquad \text{[B6]}$$

Furthermore,

$$H(0) = \frac{u_0^2 \beta \tau_{\rm L}}{(\beta \tau_{\rm L} + 1)}. \qquad \text{[B7]}$$